Propagation in Rectangular Waveguide Containing Inhomogeneous, Anisotropic Dielectric

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Summary—**The Wentzel, Kramers and Brillouin (WKB) approximation is used to solve the wave equations for propagation of guided waves in rectangular waveguide containing an inhomogeneous dielectric. The simplest form of anisotropy is used to charactertize the relative dielectric constant, *i.e.*, it is assumed that the relative permittivity tensor is diagonalized with respect to the waveguide coordinants. Each of the elements of the relative permittivity tensor is allowed to vary continuously across the broad dimension of the waveguide. The TE_{nm} and TM_{nm} cases are analyzed for the instance of completely filled guide, while the TE_{no} modes are considered for slab-loaded guide.**

I. Introduction

THE WKB APPROXIMATION[1] for solving the Schroedinger equation has been shown to be useful for solving electromagnetic wave equations.[2–6] The purpose of this work is to present a WKB analysis of wave propagation in rectangular waveguide containing an inhomogeneous, anisotropic dielectric. Losses in the dielectric may be included by allowing the elements of the relative permittivity tensor to become complex.

The rectangular waveguide is assumed to be of width a and height b, where $b \leqq a$. In the coordinant system used, x measures distance across the broad dimension of the guide, y measures distance across the short dimension and z measures distance parallel to the axis of the guide. It is assumed that the relative permittivity tensor has only diagonal elements which are given by $K_x(x)$, $K_y(x)$ and $K_z(x)$, functions of x only.

In Section II, the transcendental equation for the propagation constant and the WKB solutions for the electric and magnetic field components are given for TE_{nm}-mode propagation in wholly filled waveguide. TM_{nm}-mode propagation in wholly filled guide is considered in Section III. In Section IV, TE_{n0}-mode propagation is considered for inhomogeneously filled waveguide. Relatively simple expressions for the propagation constants result for the special case of a symmetric slab. In the Appendix, the WKB solution for a generalized wave equation is discussed. It is to be noted that the functions $\psi(x)$, $\alpha(x)$ and $\beta(x)$, which are used in the Appendix, are given different definitions in Sections II, III and IV.

In all cases, the time variation of the field components is assumed to be given by exp $(j\omega t)$ while the z dependence is given by exp $(-\Gamma z)$, where ω is the angular frequency and Γ is the propagation constant.

II. TE_{nm} Mode Propagation in Wholly Filled Guide

For this case a wave equation for H_z may be found, the other field components then being given by

$$H_x = \frac{-\Gamma}{\Gamma^2 + k_0{}^2 K_y(x)} \cdot \frac{\partial H_z}{\partial x}, \tag{1a}$$

$$H_y = \frac{-\Gamma}{\Gamma^2 + k_0{}^2 K_x(x)} \cdot \frac{\partial H_z}{\partial y}, \tag{1b}$$

$$E_x = \frac{-j\omega\mu_0}{\Gamma^2 + k_0{}^2 K_x(x)} \cdot \frac{\partial H_z}{\partial y} \tag{1c}$$

and

$$E_y = \frac{j\omega\mu_0}{\Gamma^2 + k_0{}^2 K_y(x)} \cdot \frac{\partial H_z}{\partial x}, \tag{1d}$$

where μ_0 is the permeability of vacuum, ϵ_0 is the permittivity of vacuum and $k_0^2 = \omega^2\mu_0\epsilon_0$.

Considering now (20a) in the Appendix, $\psi(x)$ is the x variation of H_z and

$$\alpha(x) = \frac{1}{\Gamma^2 + k_0{}^2 K_y(x)}, \tag{2}$$

$$\beta^2(x) = \frac{\Gamma^2 + k_0{}^2 K_y(x)}{\Gamma^2 + k_0{}^2 K_x(x)} \cdot \left[\Gamma^2 + k_0{}^2 K_z(x) - \left(\frac{m\pi}{b}\right)^2\right] \tag{3}$$

and

$$f(x) = \int_0^x \beta(x) dx. \tag{4}$$

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[1] L. I. Schiff, "Quantum Mechanics," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 184–194; 1955.

[2] J. H. Richmond, "The WKB solution for transmission through inhomogeneous plane layers," IRE Trans. on Antennas and Propagation (*Correspondence*), vol. AP-10, pp. 472–473; July, 1962.

[3] J. H. Richmond, "Propagation of surface waves on an inhomogeneous plane layer," IRE Trans. on Microwave Theory and Techniques, vol. MTT-10, pp. 554–558; November, 1962.

[4] D. A. Holmes, "A Refinement of the Theory for a Microwave Technique for Measuring Semiconductor Lifetime," Westinghouse Electric Corp., Pittsburgh, Pa., Scientific Paper No. 63-144-258-P1; July, 1963.

[5] D. A. Holmes, "Transmission and reflection properties of an inhomogeneous semiconductor slab in rectangular waveguide," Proc. IEEE (*Correspondence*), vol. 52, pp. 201–202; February 1964.

[6] D. A. Holmes, "TM and TE mode surface waves on grounded, anisotropic inhomogeneous, lossless, dielectric slabs," IEEE Trans. on Microwave Theory and Techniques (*Correspondence*), vol. MTT-12, p. 141; January, 1964.

The complete set of TE_{nm} field solutions is then given by

$$H_z = \frac{H \cos(m\pi y/b) \cos f(x)}{\sqrt{\beta(x)\alpha(x)}}, \tag{5a}$$

$$H_x = \frac{+H\Gamma \cos(m\pi y/b) \sqrt{\beta(x)} \sin f(x)}{\sqrt{\alpha(x)}\,(\Gamma^2 + k_0^2 K_y(x))}, \tag{5b}$$

$$H_y = \frac{+H\Gamma m\pi \sin(m\pi y/b) \cos f(x)}{b\sqrt{\beta(x)\alpha(x)}\,(\Gamma^2 + k_0^2 K_x(x))}, \tag{5c}$$

$$E_x = \frac{j\omega\mu_0 H m\pi \sin(m\pi y/b) \cos f(x)}{b\sqrt{\beta(x)\alpha(x)}\,(\Gamma^2 + k_0^2 K_x(x))} \tag{5d}$$

and

$$E_y = \frac{-j\omega\mu_0 H \cos(m\pi y/b) \sqrt{\beta(x)} \sin f(x)}{\sqrt{\alpha(x)}\,(\Gamma^2 + k_0^2 K_y(x))}, \tag{5e}$$

where H is merely a multiplicative constant and m is an integer.

The TE_{nm} propagation constants are determined from

$$\int_0^a \left\{\frac{\Gamma^2 + k_0^2 K_y(x)}{\Gamma^2 + k_0^2 K_x(x)} \cdot \left[\Gamma^2 + k_0^2 K_x(x) - \left(\frac{m\pi}{b}\right)^2\right]\right\}^{1/2} dx = n\pi, \tag{6}$$

where n is a nonzero integer.

III. TM_{nm} Mode Propagation in Wholly Filled Waveguide

In (20a) $\psi(x)$ is the x variation of E_z and

$$\alpha(x) = \frac{K_x(x)}{\Gamma^2 + k_0^2 K_x(x)}, \tag{7}$$

$$\beta^2(x) = \frac{K_y(x)}{K_x(x)} \cdot \frac{\Gamma^2 + k_0^2 K_x(x)}{\Gamma^2 + k_0^2 K_y(x)} \cdot \left[\frac{K_z(x)}{K_y(x)}(\Gamma^2 + k_0^2 K_y(x)) - \left(\frac{m\pi}{b}\right)^2\right] \tag{8}$$

and

$$f(x) \equiv \int_0^x \beta(x)dx. \tag{9}$$

The complete set of field solutions is

$$E_z = \frac{E \sin(m\pi y/b) \sin f(x)}{\sqrt{\beta(x)\alpha(x)}}, \tag{10a}$$

$$E_x = \frac{-\Gamma \dfrac{\partial E_z}{\partial x}}{\Gamma^2 + k_0^2 K_x(x)} = \frac{-E\Gamma\sqrt{\beta(x)} \sin(m\pi y/b) \cos f(x)}{\sqrt{\alpha(x)}\,(\Gamma^2 + k_0^2 K_x(x))}, \tag{10b}$$

$$E_y = \frac{-\Gamma \dfrac{\partial E_z}{\partial y}}{\Gamma^2 + k_0^2 K_y(x)} = \frac{-\Gamma E m\pi \cos(m\pi y/b) \sin f(x)}{b\sqrt{\beta(x)\alpha(x)}\,(\Gamma^2 + k_0^2 K_y(x))}, \tag{10c}$$

$$H_x = \frac{j\omega\epsilon_0 K_y(x) \dfrac{\partial E_z}{\partial y}}{\Gamma^2 + k_0^2 K_y(x)} = \frac{j\omega\epsilon_0 K_y(x) E m\pi \cos(m\pi y/b) \sin f(x)}{b\sqrt{\beta(x)\alpha(x)}\,(\Gamma^2 + k_0^2 K_y(x))} \tag{10d}$$

and

$$H_y = \frac{-j\omega\epsilon_0 K_x(x) \dfrac{\partial E_z}{\partial x}}{\Gamma^2 + k_0^2 K_x(x)} = \frac{-j\omega\epsilon_0 K_x(x) E\sqrt{\beta(x)} \sin(m\pi y/b) \cos f(x)}{\sqrt{\alpha(x)}\,(\Gamma^2 + k_0^2 K_x(x))} \cdot \tag{10e}$$

The TM_{nm} propagation constant is determined from

$$\int_0^a \left\{\frac{K_y(x)}{K_x(x)} \cdot \frac{\Gamma^2 + k_0^2 K_x(x)}{\Gamma^2 + k_0^2 K_y(x)} \cdot \left[\frac{K_z(x)}{K_y(x)}(\Gamma^2 + k_0^2 K_y(x)) - \left(\frac{m\pi}{b}\right)^2\right]\right\}^{1/2} dx = n\pi. \tag{11}$$

IV. TE_{n0} Mode Propagation in Slab Loaded Guide

It is now considered that a dielectric slab occupies the region from x_1 to x_2 while the rest of the waveguide is filled with air. The air-filled region from 0 to x_1 is termed I, the dielectric-filled region from x_1 to x_2 is termed II and the air-filled region from x_2 to a is denoted by III. In this section, only the TE modes with no y variation shall be considered; thus the only field components are E_y, H_x and H_z. For this case, $\psi(x)$ is the x variation of E_y when $\alpha(x) = 1$ and

$$\beta^2(x) = \Gamma^2 + k_0^2 K_y(x) \tag{12}$$

and

$$f(x, x_1) \equiv \int_{x_1}^x \beta(x)dx. \tag{13}$$

The appropriate solutions are

$$E_y = \begin{cases} E \sin(px) \quad \text{in I}, & (14a) \\ \dfrac{1}{\sqrt{\beta(x)}}\{B \sin[f(x, x_1)] + C \sin[f(x_2, x_1) - f(x, x_1)]\} \quad \text{in II}, & (14b) \\ D \sin[p(a - x)] \quad \text{in III}, & (14c) \end{cases}$$

where $p^2 = \Gamma^2 + k_0^2$ and E, B, C and D are constants.

By forcing E_y and $\partial E_y/\partial x$ to be continuous at the boundaries $x=x_1$ and $x=x_2$, the following system of equations is obtained:

$$E\sqrt{\beta(x_1)}\sin(px_1) - C\sin[f(x_2, x_1)] = 0,$$
$$Ep\cos(px_1) - \sqrt{\beta(x_1)}\{B - C\cos[f(x_2, x_1)]\} = 0,$$
$$D\sqrt{\beta(x_2)}\sin[p(a-x_2)] - B\sin[f(x_2, x_1)] = 0,$$
$$Dp\cos[p(a-x_2)] + \sqrt{\beta(x_2)} \cdot \{B\cos[f(x_2, x_1)] - C\} = 0. \quad (15)$$

Setting the determinant of (15) equal to zero, a transcendental equation for the propagation constant Γ is obtained,

$$\beta(x_1)\beta(x_2)\tan[f(x_2, x_1)]\tan(px_1)\tan[p(a-x_2)] - p^2\tan[f(x_2, x_1)] - p\beta(x_1)\tan(px_1) - p\beta(x_2)\tan[p(a-x_2)] = 0. \quad (16)$$

Eq. (16) can be greatly simplified for a centered, symmetric slab. For this case, let $x_1=a-x_2=d$ and $\beta(x)=\beta(a-x)$ for $x_1 \leqq x \leqq x_2$. It then follows that $\tan[f(x_2, x_1)] = \tan[2f(a/2, d)]$. Using the relation

$$\tan[2f(a/2, d)] = \frac{2\tan[f(a/2, d)]}{1-\tan^2[f(a/2, d)]}$$

in (16), one obtains, after factoring,

$$\beta(d)\tan(pd)\tan[f(a/2, d)] - p = 0 \quad (17)$$

and

$$\beta(d)\tan(pd) + p\tan[f(a/2, d)] = 0. \quad (18)$$

For arbitrary x_1, x_2 and $\beta(x)$, the constants B, C and D can be found in terms of E from (15). For the centered, symmetric slab the solutions of (17) and (18) can be shown to correspond to symmetrical and asymmetrical modes, respectively. In general

$$C = \frac{E\sqrt{\beta(x_1)}\sin(px_1)}{\sin[f(x_2, x_1)]},$$
$$B = \frac{E\sin(px_1)}{\sqrt{\beta(x_1)}}\{p\cot(px_1) + \beta(x_1)\cot[f(x_2, x_1)]\}$$

and

$$D = \frac{E\sin(px_1)\sin[f(x_2, x_1)]}{\sqrt{\beta(x_1)\beta(x_2)}\sin[p(a-x_2)]} \cdot \{p\cot(px_1) + \beta(x_1)\cot[f(x_2, x_1)]\}.$$

The equation for D for the centered, symmetric slab reduces to

$$D = \frac{E\sin[2f(a/2, d)]}{\beta(d)} \cdot \{p\cot(pd) + \beta(d)\cot[2f(a/2, d)]\}. \quad (19)$$

By substituting (17) into (19) one finds that $D=E$; by substituting (18) into (19) one finds that $D=-E$. Therefore, the propagation constants found from (17) are those for which E_y is symmetric with respect to the $x=a/2$ plane. The asymmetric modes result from the solutions of (18).

V. Discussion and Conclusion

WKB techniques have been shown to be useful for analytically studying the propagation characteristics of infinite rectangular waveguide filled with an anisotropic material. The off-diagonal elements of the relative permittivity tensor have been set equal to zero while the diagonal elements have been considered to be slowly varying functions of distance across the broad dimension of the waveguide. The complete sets of field solutions and the eigenvalue equations for the propagation constants have been determined for both TE_{nm} and TM_{nm} mode propagation.

For waveguide loaded with an inhomogeneous, anisotropic slab, the TE_{n0} modes have been considered. The transcendental equation for the propagation constants for a centered, symmetrical slab was factored into two equations corresponding to the symmetrical and asymmetrical modes.

Dr. Richmond has numerically shown the validity of the WKB approximation for electromagnetic problems in his earlier works,[2,3] therefore no numerical work has been presented herein. The introduction of numerical work would necessitate assumptions concerning the exact functional behavior of the dielectric constant tensor elements.

Appendix

Consider the wave equation

$$\frac{1}{\alpha(x)}\cdot\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial\psi(x)}{\partial x}\right) + \beta^2(x)\psi(x) = 0, \quad (20a)$$

where $\psi(x)$ is the x variation of some field component and $\alpha(x)$ and $\beta(x)$ are slowly varying functions of x. By substituting

$$\psi(x) = \psi_0\exp\{j\omega S(x)\}$$

into (20a), choosing a series expansion for $S(x)$ of the form

$$S(x) = \sum_{n=1}^{\infty}\frac{S_n(x)}{\omega^n},$$

and then following the technique outlined in Schiff,[1] the resulting approximate solution for $\psi(x)$ is

$$\psi(x) = \frac{\psi_0\exp\left\{\pm j\int^x \beta(x)dx\right\}}{\sqrt{\beta(x)\alpha(x)}}. \quad (20b)$$

Eq. (20b) can alternatively be written as

$$\psi(x) = \frac{\psi_0 \begin{matrix}\sin\\ \cos\end{matrix} \left\{ \int^x \beta(x) dx \right\}}{\sqrt{\beta(x)\alpha(x)}} \cdot \qquad (20c)$$

When differentiating the solutions (20b) and (20c), it is conventional[7] to consider that the denominator varies much more slowly than the numerator so that

$$\frac{\partial \psi(x)}{\partial x} = \frac{\pm j\psi_0 \sqrt{\beta(x)} \exp\left\{ \pm j \int^x \beta(x) dx \right\}}{\sqrt{\alpha(x)}} \cdot \qquad (20d)$$

[7] L. M. Brekhovskikh, "Waves in Layered Media," Academic Press, Inc., New York, N. Y., p. 196; 1960.

The solutions (20b) and (20c) are valid only when $\alpha(x)$ and $\beta(x)$ satisfy the condition

$$\left| \frac{\dfrac{\partial S_1(x)}{\partial x}}{\omega \dfrac{\partial S_0(x)}{\partial x}} \right| = \left| \frac{\dfrac{1}{\beta(x)} \dfrac{\partial \beta(x)}{\partial x} + \dfrac{1}{\alpha(x)} \dfrac{\partial \alpha(x)}{\partial x}}{2\beta(x)} \right| \ll 1. \qquad (20e)$$

All of the solutions given in Sections II, III and IV must satisfy condition (20e).

Thin-Film Waveguide Bolometers for Multimode Power Measurement

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Summary—**Thin-film bolometers have been developed for measuring the total (unwanted) power that could be transmitted in any or all possible modes and at many frequencies above the normal operating band.**

The bolometer is a thin metal film which is placed so that it intercepts all the power flowing down the waveguide. When the power in the fundamental frequency is filtered out and only power at higher frequencies remains in the waveguide containing the bolometer, then it can be used to measure the total spurious power emitted by a high-power transmitter above its fundamental frequency band. Measurements have been made up to 15 Gc in *S*-band waveguide.

A variety of materials and shapes were tested and the bolometers were shown to be capable of measuring equally well several different modes and frequencies separately and in combination.

I. Introduction

A FIRST STEP in reducing RFI emission from a microwave transmitter is the accurate determination of the total power in all the undesired frequency components traveling in the waveguide transmission line. A power meter that operates over an extremely broad band and maintains uniform sensitivity for all modes that might exist within the measurement band would be a very useful RFI monitoring device. This conclusion was reached after examining the facts and arguing from them as follows.

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A harmonic sampler [1] had been designed earlier at Stanford Research Institute, and further developed at Airborne Instruments Laboratory, Deer Park, N. Y. [2]. Such a device has been used to measure the spectral output of a high-power source from the second to the sixth harmonic [3]. This versatile instrument is a relatively complicated device. It was felt that a simpler instrument, which would measure only the *total* spurious output from a high-power source (without giving the spectral distribution), would be a most useful adjunct to a high-power system when it is required to minimize the spurious-frequency output. This conclusion was based partly on the following measured result [3]: In the process of adjusting the electrode voltages of a high-power klystron, it was found that minimizing the second harmonic tended to minimize all the other harmonics also. It will of course require more measurements to determine whether this result ordinarily holds. In the meantime, however, it is suggested that a measurement of the single quantity, the *total spurious power* (without regard to spectral distribution), may enable one to set the control voltages to minimize any particular RFI. (Furthermore it might also indicate the approximate power level at any particular frequency, if the spectral distribution is known beforehand.) The reader should